Available at WWW.MATHEMATICSWEB.ORG
POWERED BY SCIENCE @ DIRECT®
JOURNAL OF Algebra
Journal of Algebra 267 (2003) 635–653
www.elsevier.com/locate/jalgebra

# The finite basis problem for quasivarieties and pseudovarieties generated by regular semigroups
# I. Quasivarieties generated by regular semigroups

D.F. Cowan,[a] N.R. Reilly,[b,*] P.G. Trotter,[c] and M.V. Volkov[d]

[a] *Department of Mathematics and Statistics, University of Lethbridge, Lethbridge, Alberta, Canada T1K 3M4*
[b] *Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6*
[c] *School of Mathematics and Physics, University of Tasmania, Hobart, Tasmania, Australia 7001*
[d] *Department of Mathematics and Mechanics, Ural State University, Ekaterinburg, Russia 620083*

Received 23 August 2001

Communicated by T.E. Hall

## Introduction

This is the first part of a series of three papers in which we show that certain important classes of regular semigroups admit no finite basis for their quasiidentities and pseudoidentities. All our results arise as applications of a general technique that extends an approach recently invented by Higgins and Margolis [6] for a different purpose. The aim of the present paper is twofold: to start with, we introduce our main construction and study some of its basic properties, and then, as the first application, we demonstrate how this construction can be used when studying the finite basis problem for quasivarieties generated by regular semigroups. The two other papers in the series will contain several applications of our technique to the finite basis problem for semigroup pseudovarieties. In the second paper we consider pseudovarieties generated by classes of regular semigroups which appear to be natural from the standpoint of the quickly progressing theory of so-called $e$-varieties of regular semigroups (from inverse and orthodox up to locally $E$-solid), while in the third paper the role of “regular generators” is played by some important semigroups of order preserving mappings on a finite chain.

We refer the reader to the books by Clifford and Preston [2] and Howie [7] for a general introduction to semigroup theory, to the books by Petrich [12] and Gorbunov [3]

* Corresponding author.
*E-mail address:* nreilly@cs.sfu.ca (N.R. Reilly).


doi:10.1016/S0021-8693(03)00394-6

for information regarding inverse semigroups and quasivarieties, respectively, and to the survey paper [15] by the third-named author for an overview of the theory of $e$-varieties of regular semigroups.

## 1. The main construction and its basic properties

We start with an arbitrary family $\{b_k\}_{k\in K}$ of partial injective mappings on a set $X$. Let $U$ be the semigroup generated by the injections $b_k$, $k \in K$. In [6, Section 3], Higgins and Margolis have constructed (provided that the sets $X$ and $K$ are finite) a combinatorial finite semigroup $S(U)$ with commuting idempotents such that if $S(U)$ divides some finite inverse semigroup $I$ then $U$ divides $I$ also. We are going to extend their construction in order to obtain a sequence $S_m(U)$, $m = 1, 2, \dots,$ of semigroups with similar properties. Though it follows the ideas of [6], our construction (provided $|K| > 1$) is not a straightforward generalization of that by Higgins and Margolis; we also drop the restriction that the sets $X$ and $K$ are finite as this is not essential for the results collected in the present paper. (We shall return to the finite case in the second paper of our series in which we shall extend the aforementioned division property to a larger class of regular semigroups.)

We observe that the notation $S_m(U)$ (as well as the notation $S(U)$ in [6]) is not completely justified because, as the reader will see, the semigroup $S_m(U)$ depends on the choice of the generators of the semigroup $U$. This should not cause any confusion as long as a generating system of $U$ is explicitly specified.

For a partial mapping $\alpha$ on a set, we denote by $\operatorname{dom}\alpha$ its *domain*, that is, the subset where the mapping $\alpha$ is defined, and by $\operatorname{ran}\alpha$ its *range*.

Now, having fixed the set $X$, partial injections $b_k$, $k \in K$, and a positive integer $m$, consider a disjoint copy $X' = \{x' \mid x \in X\}$ of $X$ and form the following set of triples:

$$T_{m,K}(X) = \bigl(X \cup X'\bigr) \times \{0, 1, \dots, m\} \times K.$$

Then, for each $x \in X$, we identify the triples $[x, 0, k]$ for all $k \in K$. Speaking more formally, we factor the set $T_{m,K}(X)$ over the equivalence whose non-singleton classes are of the form $\{[x, 0, k] \mid k \in K\}$; it will be convenient to denote such a class simply by $x$ thus identifying the collection of all these classes with the initial set $X$. The quotient set $\overline{T}_{m,K}(X)$ may then be thought as the disjoint union of $X$ with the sets of triples

$$X_{j,k} = \bigl\{[x, j, k] \mid x \in X\bigr\}, \quad j \in \{1, \dots, m\},\ k \in K,$$

and

$$X'_{j,k} = \bigl\{\bigl[x', j, k\bigr] \mid x' \in X'\bigr\}, \quad j \in \{0, 1, \dots, m\},\ k \in K.$$

The semigroup $S_m(U)$ is a subsemigroup of the symmetric inverse semigroup consisting of partial injective mappings on the set $\overline{T}_{m,K}(X)$. We start the list of elements of the semigroup $S_m(U)$ with mappings whose role is to encode the injections $b_k$. Namely, for each

$k \in K$, we denote by $\beta_k$ the mapping whose domain is $\operatorname{dom} b_k$ and whose action is defined by the rule:

$$x\beta_k = \left[(xb_k)', 0, k\right] \in X'_{0,k} \quad \text{for each } x \in \operatorname{dom} b_k.$$

Further, for each $k \in K$ and for each $j = 1, 2, \ldots, m$, we consider the mapping $\gamma_{j,k}$ which is defined on the set $X_{j,k}$ and maps it onto the set $X'_{j,k}$ by "priming" the first component of the triple:

$$[x, j, k]\gamma_{j,k} = \left[x', j, k\right] \quad \text{for each } x \in X.$$

Another sequence of partial injections, also defined on the sets $X_{j,k}$ for each $k \in K$ and for each $j = 1, 2, \ldots, m$, consists of the transformations $\delta_{j,k}$ which map $X_{j,k}$ in a way similar to that of $\gamma_{j,k}$, but onto the "previous" set $X'_{j-1,k}$:

$$[x, j, k]\delta_{j,k} = \left[x', j-1, k\right] \quad \text{for each } x \in X.$$

We augment the sequence by adding, for each $k \in K$, the mappings $\delta_{0,k}$ which are all defined on the set $X$ and act as follows:

$$x\delta_{0,k} = \left[x', m, k\right] \in X'_{m,k} \quad \text{for each } x \in X.$$

Figure 1 schematically shows the action of the mappings $\beta_k, \gamma_{j,k}, \delta_{j,k} \in S_m(U)$ for the case $m = 3$ and $K = \{1, 2\}$. The diagram can be viewed as $(|K| = )$ 2 stars with $(m+1 = )$ 4 vertices hinged at $X$. The general case has $|K|$ stars, each with $m + 1$ vertices and all hinged together at $X$.

Clearly, the mappings $\beta_k, \gamma_{j,k}, \delta_{j,k}$ generate a null semigroup, as domains and ranges are disjoint. Define the semigroup $S_m(U)$ to be the union of this null semigroup with the combinatorial Brandt ideal $B$ consisting of all mappings between singleton subsets in $\overline{T}_{m,K}(X)$ and the zero $\varepsilon$ (the nowhere defined map). Since, by the construction, the only group elements in $S_m(U)$ are the idempotents of $B$, we immediately obtain

**Proposition 1.1.** *The semigroup $S_m(U)$ is combinatorial and has commuting idempotents.*

The following property of the semigroup $S_m(U)$ also is easy to verify:

**Proposition 1.2.** *The Rees congruence corresponding to the ideal $B$ is the least non-trivial congruence on the semigroup $S_m(U)$.*

**Proof.** The ideal $B$ is a combinatorial Brandt semigroup and as such it is known to be congruence-free, that is, every non-trivial congruence on $B$ coincides with the universal relation. Therefore, it suffices to show that the restriction of an arbitrary non-trivial congruence $\sigma$ on $S_m(U)$ to the ideal $B$ is a non-trivial congruence on $B$. Thus, let $\alpha_1, \alpha_2$ be two distinct elements in $S_m(U)$ such that $(\alpha_1, \alpha_2) \in \sigma$. By the construction of $S_m(U)$, if $\alpha_1 \neq \alpha_2$, then either the domains or the ranges of these two injections must be different. By

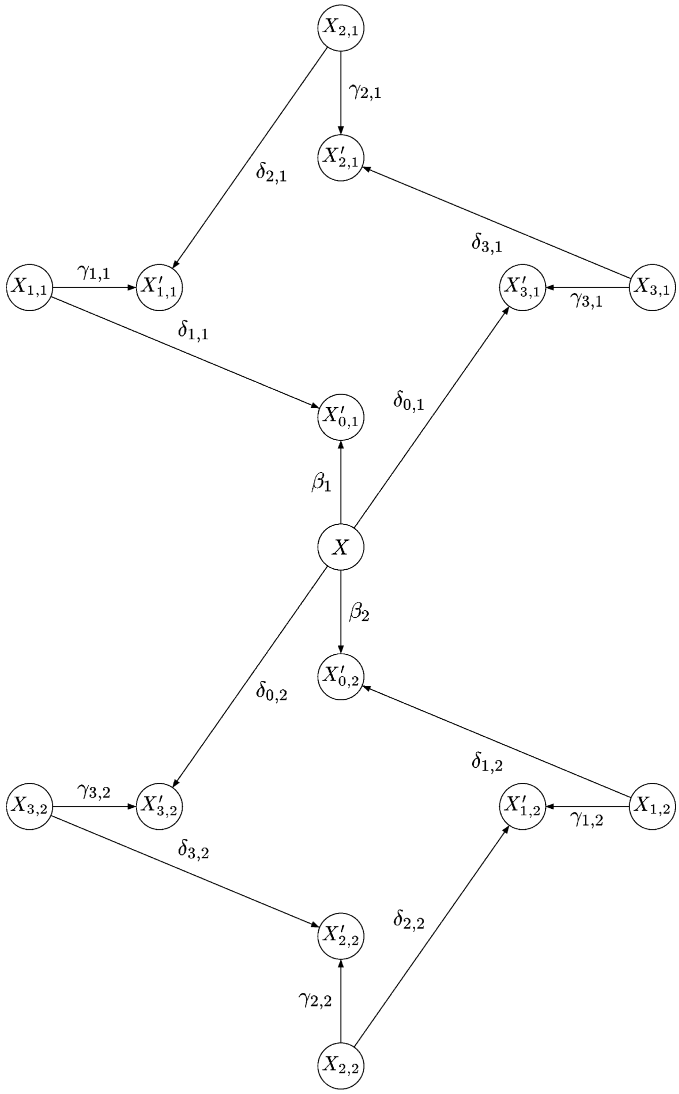


Fig. 1. Domains and ranges of the elements in $S_3(U) \setminus B$ ($K = \{1, 2\}$).

symmetry, we may assume that $\operatorname{dom}\alpha_1 \neq \operatorname{dom}\alpha_2$ and that there exists $t \in \operatorname{dom}\alpha_1 \setminus \operatorname{dom}\alpha_2$. Consider the mapping $\tau \in B$ whose domain and range coincide with the singleton $\{t\}$. Then $\tau_1 = \tau\alpha_1$ is a non-zero element in $B$, while $\tau\alpha_2 = \varepsilon$, and, of course, $(\tau_1, \varepsilon) \in \sigma$. We see that $\sigma$ restricted to $B$ remains non-trivial, as required. $\square$

**Corollary 1.3.** *The semigroup $S_m(U)$ is subdirectly irreducible.*

Now we approach a more involved property of $S_m(U)$ which is crucial for the present paper. In order to formulate this property, we recall that $U$ consists of partial injective mappings on the set $X$, and thus, is contained in the symmetric inverse semigroup $\mathcal{I}_X$ on $X$. Let $J(U)$ denote the *inverse hull* of $U$, that is, the inverse subsemigroup of $\mathcal{I}_X$ generated by $U$. Clearly, as a plain semigroup $J(U)$ is generated by the injections $b_k$, $k \in K$, and their inverses in $\mathcal{I}_X$. By an *inverse divisor* of an inverse semigroup $I$ we mean a homomorphic image of an inverse subsemigroup of $I$.

**Proposition 1.4.** *If the semigroup $S_m(U)$ embeds in an inverse semigroup $I$, then $J(U)$ is an inverse divisor of $I$.*

**Proof.** We identify $S_m(U)$ with its image in $I$. Clearly, we may assume that $I$ as an inverse semigroup is generated by $S_m(U)$, in other words, that $I$ consists of products of elements of $S_m(U)$ and their inverses. Then the Brandt ideal $B$ of $S_m(U)$ is easily seen to be an ideal in $I$ as well, and the zero $\varepsilon$ of $S_m(U)$ also serves as a zero in $I$.

Fix an element $x_0 \in X$ and for each $t \in \overline{T}_{m,K}(X)$, consider the mapping $\xi(t) \in B$ which sends $x_0$ to $t$. Then for each $\alpha \in S_m(U)$,

$$\xi(t)\alpha = \begin{cases} \xi(t\alpha) & \text{if } t \in \operatorname{dom}\alpha, \\ \varepsilon & \text{if } t \notin \operatorname{dom}\alpha. \end{cases} \tag{1}$$

Thus the action of $\alpha$ (by multiplying on the right) on the set

$$\left\{\xi(t) \mid t \in \overline{T}_{m,K}(X)\right\}$$

mimics the action of $\alpha$ as a partial mapping of the set $\overline{T}_{m,K}(X)$. We are going to verify that multiplying on the right by $\alpha^{-1}$, the inverse of $\alpha$ in $I$, in a similar manner mimics the action of the inverse of the injection $\alpha$.

Indeed, let $z \in \operatorname{ran}\alpha$, that is, $z = t\alpha$ for some $t \in \operatorname{dom}\alpha$. Then by (1), $\xi(t)\alpha = \xi(z)$. If $\lambda \in B$ is the mapping that sends $z$ to $t$, then $\xi(z)\lambda = \xi(z\lambda) = \xi(t)$. Thus, $\xi(t)\alpha\lambda = \xi(t)$. We are in a position to employ the following simple property of inverse semigroups whose proof we have included for the sake of completeness:

**Lemma 1.5.** *If $a, b, c$ are elements of an inverse semigroup and $a = abc$, then $a = abb^{-1}$.*

**Proof.**

$$\begin{aligned} a &= abc = aa^{-1}abb^{-1}bc = abb^{-1}a^{-1}abc \\ &= abb^{-1}a^{-1}a = aa^{-1}abb^{-1} = abb^{-1}. \qquad \square \end{aligned}$$

By Lemma 1.5 we conclude that $\xi(t)\alpha\alpha^{-1} = \xi(t)$, whence $\xi(z)\alpha^{-1} = \xi(t)$.

Now let $z \notin \operatorname{ran}\alpha$; we aim to show that $\xi(z)\alpha^{-1} = \varepsilon$. Observe that $\xi(z)\alpha^{-1} \in B$ as $B$ is an ideal in $I$. Non-zero elements in $B$ are mappings between singleton subsets in $\overline{T}_{m,K}(X)$, and therefore, if $\xi(z)\alpha^{-1} = \mu \neq \varepsilon$, then $\mu$ maps some $s \in \overline{T}_{m,K}(X)$ to some $t \in \overline{T}_{m,K}(X)$. Multiplying on the left by the idempotent $\xi(x_0)$, we obtain $\xi(x_0)\xi(z)\alpha^{-1} = \xi(x_0)\mu$. By (1),

$$\xi(x_0)\xi(z) = \xi(z) \quad \text{and} \quad \xi(x_0)\mu = \begin{cases} \xi(t) & \text{if } s = x_0, \\ \varepsilon & \text{if } s \neq x_0. \end{cases}$$

Thus, if $\xi(z)\alpha^{-1} \neq \varepsilon$, then $\xi(z)\alpha^{-1} = \xi(t)$ for some $t \in \overline{T}_{m,K}(X)$. Multiplying the latter equality through on the right by the element $\nu \in B$ such that $t\nu = z$, we get $\xi(z)\alpha^{-1}\nu = \xi(t)\nu = \xi(z)$. By Lemma 1.5 this implies $\xi(z)\alpha^{-1}\alpha = \xi(z)$. On the other hand, by (1),

$$\xi(z)\alpha^{-1}\alpha = \xi(t)\alpha = \begin{cases} \xi(t\alpha) & \text{if } t \in \operatorname{dom}\alpha, \\ \varepsilon & \text{if } t \notin \operatorname{dom}\alpha. \end{cases}$$

We must conclude that $\xi(z) = \xi(t\alpha)$ for some $t \in \operatorname{dom}\alpha$, whence $z = t\alpha$ contradicting the assumption $z \notin \operatorname{ran}\alpha$.

Thus,

$$\xi(z)\alpha^{-1} = \begin{cases} \xi(t) & \text{if } z = t\alpha \text{ for some } t \in \operatorname{dom}\alpha, \\ \varepsilon & \text{if } z \notin \operatorname{ran}\alpha. \end{cases} \tag{2}$$

Now, for each $k \in K$, consider the following product in $I$:

$$a_k = \beta_k \delta_{1,k}^{-1}\gamma_{1,k}\delta_{2,k}^{-1}\gamma_{2,k}\cdots\delta_{m,k}^{-1}\gamma_{m,k}\delta_{0,k}^{-1}.$$

The action of $a_k$ (by multiplying on the right) on the set $\Xi = \{\xi(x) \mid x \in X\}$ mimics the action of the injection $b_k$ on the set $X$. Indeed, if $x \notin \operatorname{dom} b_k = \operatorname{dom}\beta_k$, then $\xi(x)\beta_k = \varepsilon$, whence $\xi(x)a_k = \varepsilon$. Now let $x \in \operatorname{dom} b_k$ and $xb_k = y$. Then alternatively applying the formulas (1) and (2), we may step by step calculate the product $\xi(x)a_k$ by walking (from left to right) up and down the "comb" shown in Fig. 2.

Clearly, the calculation results in the equality $\xi(x)a_k = \xi(y)$. Analogously, multiplying elements of $\Xi$ on the right by

$$a_k^{-1} = \delta_{0,k}\gamma_{m,k}^{-1}\delta_{m,k}\cdots\gamma_{2,k}^{-1}\delta_{2,k}\gamma_{1,k}^{-1}\delta_{1,k}\beta_k^{-1}$$

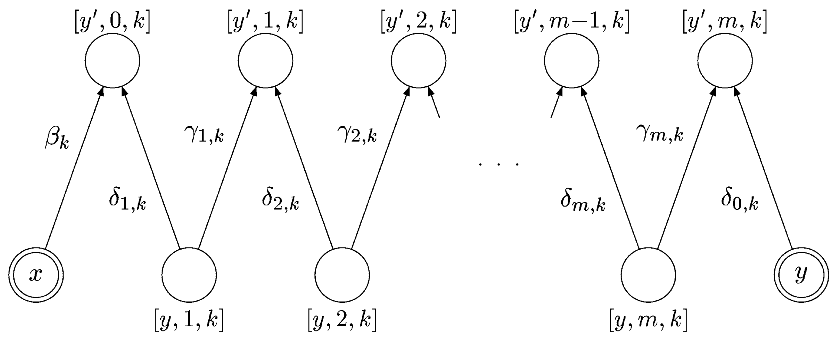


Fig. 2. Calculating $\xi(x)a_k$ and $\xi(y)a_k^{-1}$.

mimics the action of the mappings $b_k^{-1} \in \mathcal{I}_X$ on the set $X$. Indeed, if $y \in \operatorname{ran} b_k$, then walking along the zigzags in Fig. 2 from right to left and as above alternating the applications of (2) and (1), we get $\xi(y)a_k^{-1} = \xi(x)$. On the other hand, if $y \notin \operatorname{ran} b_k$, then a similar calculation shows that

$$\xi(y)\delta_{0,k}\gamma_{m,k}^{-1}\delta_{m,k}\cdots\gamma_{2,k}^{-1}\delta_{2,k}\gamma_{1,k}^{-1}\delta_{1,k} = \xi\bigl([y', 0, k]\bigr).$$

Hence $\xi(y)a_k^{-1} = \xi([y', 0, k])\beta_k^{-1}$. Since $y \notin \operatorname{ran} b_k$, we have $[y', 0, k] \notin \operatorname{ran} \beta_k$ and by (2), $\xi([y', 0, k])\beta_k^{-1} = \varepsilon$.

Now denote by $V$ the subsemigroup of $I$ generated by the elements $a_k$ and $a_k^{-1}$, $k \in K$. Clearly, $V$ is an inverse subsemigroup. Consider the partial mapping $\varphi : V \to J(U)$ that sends $a_k$ to $b_k$ and $a_k^{-1}$ to $b_k^{-1}$. In order to show that this mapping can be correctly extended to a homomorphism of $V$ onto $J(U)$, it suffices to check that for all $c_1, \dots, c_\ell \in \{a_k, a_k^{-1}\}_{k\in K}$ and for every pair $w_1, w_2$ of semigroup words, $w_1(c_1\varphi, \dots, c_\ell\varphi) = w_2(c_1\varphi, \dots, c_\ell\varphi)$ in $J(U)$ whenever $w_1(c_1, \dots, c_\ell) = w_2(c_1, \dots, c_\ell)$ in $V$. Assume the latter holds, then for each $x \in X$,

$$\xi(x)w_1(c_1, \dots, c_\ell) = \xi(x)w_2(c_1, \dots, c_\ell). \tag{3}$$

As we have shown that for all $x \in X$ and for all $k \in K$,

$$\xi(x)a_k^{\pm 1} = \begin{cases} \xi(xb_k^{\pm 1}) & \text{if } x \in \operatorname{dom} b_k^{\pm 1}, \\ \varepsilon & \text{if } x \notin \operatorname{dom} b_k^{\pm 1}, \end{cases}$$

the equality (3) ensures that the maps $w_1(c_1\varphi, \dots, c_\ell\varphi)$ and $w_2(c_1\varphi, \dots, c_\ell\varphi)$ share the same domain and that

$$\xi\bigl(xw_1(c_1\varphi, \dots, c_\ell\varphi)\bigr) = \xi\bigl(xw_2(c_1\varphi, \dots, c_\ell\varphi)\bigr)$$

for all $x$ in this common domain. Hence $w_1(c_1\varphi, \dots, c_\ell\varphi) = w_2(c_1\varphi, \dots, c_\ell\varphi)$, as required, and the semigroup $J(U)$ turns out to be a homomorphic image of an inverse subsemigroup of $I$. $\square$

## 2. Subsemigroups of the semigroup $S_m(U)$

As in the previous section, we start with a semigroup $U$ generated by a family $\{b_k\}_{k\in K}$ of partial injections of a set $X$. For a technical reason (see the proof of Lemma 2.1 below), here we shall additionally assume that each $b_k$ is *extendable*, that is, can be extended to a bijection of $X$. This is automatically true if the set $X$ is finite. If $X$ is infinite and it is not true for each $b_k$ as given, we can replace $X$ by $X \cup X^\sharp$ where $X^\sharp$ is disjoint from $X$ and has the same cardinality and treat the $b_k$ as partial injections on this enlarged set. Clearly, each $b_k$ then becomes extendable while the semigroup $U$ generated by the $b_k$, $k \in K$, remains—up to isomorphism—the same. Thus, our assumption does not lead to any loss of generality.

Now take a positive integer $m$ and construct the semigroup $S_m(U)$ as described in Section 1. Here we are going to discuss some important properties of $m$-generated subsemigroups in $S_m(U)$. Let $S$ be such a subsemigroup. By the construction of $S_m(U)$, the mappings $\gamma_{j,k}$ and $\delta_{j,k}$ $(1 \leqslant j \leqslant m,\ k \in K)$ are indecomposable in $S_m(U)$, whence $S$ can contain at most $m$ of them. Thus, for every fixed $k$, at least one of the $2m$ mappings $\gamma_{j,k}$ and $\delta_{j,k}$ is absent from $S$. (Informally, it means that "cycles" formed, as in Fig. 1, by the $\beta_k, \delta_{1,k}, \gamma_{1,k}, \delta_{2,k}, \gamma_{2,k}, \ldots, \delta_{m,k}, \gamma_{m,k}, \delta_{0,k}$ are broken in the subsemigroup $S$.) Hence $S$ is contained in the subsemigroup $S\{\zeta_k\}$ of $S_m(U)$ which we get by omitting, for each $k \in K$, a certain mapping

$$\zeta_k \in \{\gamma_{j,k}, \delta_{j,k},\ j = 1, 2, \ldots, m\}$$

from the generating system of $S_m(U)$. We note that every subsemigroup of the form $S\{\zeta_k\}$ contains the ideal $B$ of $S_m(U)$ (recall that $B$ consists of all mappings between singleton subsets in $\overline{T}_{m,K}(X)$ and the zero mapping $\varepsilon$) as well as all the mappings $\beta_k$ and $\delta_{0,k}$, $k \in K$. Figure 3 schematically represents a typical subsemigroup of this kind.

Since we are interested in properties inherited by subsemigroups, we may focus on a few large but uniformly organized subsemigroups $S\{\zeta_k\}$ rather than try to control the too numerous and too diverse collection of $m$-generated subsemigroups in $S_m(U)$. In fact, we shall study even larger semigroups, namely, for each semigroup $S\{\zeta_k\}$, we consider its inverse hull $J(S\{\zeta_k\})$ in the symmetric inverse semigroup on the set $\overline{T}_{m,K}(X)$. Recall that each element in $J(S\{\zeta_k\})$ is a product of maps from $S\{\zeta_k\}$ and their inverses.

**Lemma 2.1.** *The set $\overline{T}_{m,K}(X)$ can be well-ordered in such a way that all mappings in $J(S\{\zeta_k\})$ become order preserving.*

**Proof.** Since mappings contained in $B$, the combinatorial Brandt ideal of $J(S\{\zeta_k\})$, preserve every order and since the inverse of an order preserving injection and the product of order preserving mappings are order preserving, it suffices to construct a well-ordering respected by all $\beta_k$, $\gamma_{j,k}$, $\delta_{j,k}$ except $\zeta_k$. For each $k \in K$, fix an extension of the mapping $\beta_k$ to a bijection $\overline{\beta}_k : X \to X'_{0,k}$—such an extension exists by the assumption made at the beginning of this section that each injection $b_k$ extends to a bijection of the set $X$ and the cardinalities of $X$ and $X'_{0,k}$ are equal. Now fix an arbitrary well-ordering $\preccurlyeq$ of the set $X$ and transfer it to the set $X'_{0,k}$ by letting

$$x\overline{\beta}_k \preccurlyeq y\overline{\beta}_k \text{ in } X'_{0,k} \quad \text{if and only if} \quad x \preccurlyeq y \text{ in } X.$$

This clearly makes the injection $\beta_k$ order preserving independently of how the order extends to the rest of the set $\overline{T}_{m,K}(X)$. If $\zeta_k \neq \delta_{1,k}$, we transfer the order from $X'_{0,k}$ to $X_{1,k}$ by letting

$$[x, 1, k] \preccurlyeq [y, 1, k] \text{ in } X_{1,k} \quad \text{if and only if} \quad \left[x', 0, k\right] \preccurlyeq \left[y', 0, k\right] \text{ in } X'_{0,k}.$$

Then $\delta_{1,k}$ becomes order preserving. If $\zeta_k \neq \gamma_{1,k}$, we transfer the order further, from $X_{1,k}$ to $X'_{1,k}$ and so on, in a similar way. We continue the process "clockwise" (with respect to

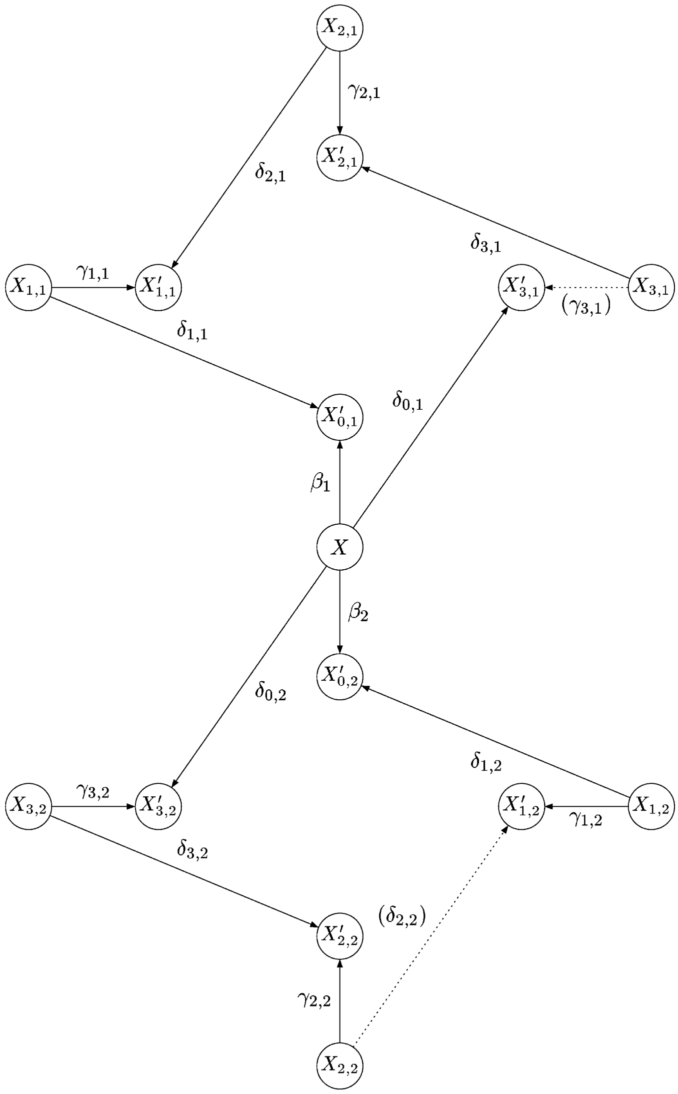


Fig. 3. A scheme of the subsemigroup $S\{\gamma_{3,1}, \delta_{2,2}\}$ ($K = \{1, 2\}, m = 3$).

the orientation as chosen in Fig. 3) until we reach the excluded mapping $\zeta_k$. Then we start expanding the order in the opposite direction, first from $X$ to $X'_{m,k}$ by letting

$$[x', m, k] \preccurlyeq [y', m, k] \text{ in } X_{1,k} \quad \text{if and only if} \quad x \preccurlyeq y \text{ in } X,$$

then (provided that $\zeta_k \neq \gamma_{m,k}$) from $X'_{m,k}$ to $X_{m,k}$ and so on, until we reach $\zeta_k$ again. Finally, we get each of the sets $X_{j,k}$ and $X'_{j,k}$ well-ordered. It is then clear that choosing an arbitrary well-ordering of the set $K$, we can form the ordinal sum of $X$ with all the sets $X_{j,k}$ and $X'_{j,k}$ thus obtaining a well-ordering $\preccurlyeq$ of the whole set $\overline{T}_{m,K}(X)$. By the construction, all the mappings $\beta_k$, $\gamma_{j,k}$, $\delta_{j,k}$ except $\zeta_k$ $(k \in K)$ preserve this order. $\square$

For the rest of the section, we fix the well-ordering $\preccurlyeq$ built in the proof of Lemma 2.1. We also note that, by the construction, the well-ordered sets $X$, $X_{j,k}$, $X'_{j,k}$ are order isomorphic. Thus, for every pair of sets $Y, Z \in \{X, X_{j,k}, X'_{j,k}\}$ there exists a unique order isomorphism $\phi_{Y,Z} : Y \to Z$ between them.

By the construction of the semigroup $J(S\{\zeta_k\})$, each non-zero mapping $\alpha \in J(S\{\zeta_k\})$ has its domain within a unique set $Y \in \{X, X_{j,k}, X'_{j,k}\}$ and its range contained in a unique set $Z \in \{X, X_{j,k}, X'_{j,k}\}$, $1 \leqslant j \leqslant m$, $k \in K$. We shall denote these sets $Y$ and $Z$ by $\underline{\operatorname{dom}}\,\alpha$ and $\underline{\operatorname{ran}}\,\alpha$, respectively. The following observation is an easy consequence of the proof of Lemma 2.1:

**Corollary 2.2.** *Let $\alpha \in J(S\{\zeta_k\}) \setminus B$, $\underline{\operatorname{dom}}\,\alpha = Y$, $\underline{\operatorname{ran}}\,\alpha = Z$. Then for every $t \in \operatorname{dom}\alpha$, $t\alpha = t\phi_{Y,Z}$.*

**Proof.** By the definition of the order $\preccurlyeq$, each $\gamma_{j_k}$ (respectively, $\delta_{j,k}$) is an order isomorphism of $X_{j,k}$ onto $X'_{j,k}$ (respectively, $X'_{j-1,k}$) while $\beta_k$ extends to an isomorphism of $X$ to $X'_{0,k}$. Consequently, each $\alpha \in J(S\{\zeta_k\}) \setminus B$ extends to an isomorphism from $\underline{\operatorname{dom}}\,\alpha = Y$ to $\underline{\operatorname{ran}}\,\alpha = Z$, that is, $\alpha$ extends to $\phi_{Y,Z}$. $\square$

We need some information concerning the $\mathscr{D}$-structure of the semigroup $J(S\{\zeta_k\})$. In general, when some of the injections $b_k$ are not totally defined on $X$, this $\mathscr{D}$-structure may be rather complicated. (In the exceptional case when all $b_k$ are total, it can easily be shown that the semigroup $J(S\{\zeta_k\})$ has exactly 3 $\mathscr{D}$-classes: $\{\varepsilon\}$, $B \setminus \{\varepsilon\}$, and $J(S\{\zeta_k\}) \setminus B$.) Fortunately, for our purposes, the following simple consequence of Corollary 2.2 well suffices.

**Lemma 2.3.** *Let $\iota, \kappa$ be $\mathscr{D}$-related idempotents in $J(S\{\zeta_k\}) \setminus B$. If $\underline{\operatorname{dom}}\,\iota = \underline{\operatorname{dom}}\,\kappa$, then $\iota = \kappa$.*

**Proof.** By a well-known property of $\mathscr{D}$-related idempotents in an inverse semigroup, there exists an element $\alpha \in J(S\{\zeta_k\})$ such that $\iota = \alpha^{-1}\alpha$, $\kappa = \alpha\alpha^{-1}$. Hence $\operatorname{dom}\alpha = \operatorname{dom}\kappa$ and $\operatorname{ran}\alpha = \operatorname{ran}\iota = \operatorname{dom}\iota$. Denote $\underline{\operatorname{dom}}\,\iota = \underline{\operatorname{dom}}\,\kappa$ by $Y$. Then $\underline{\operatorname{dom}}\,\alpha = \underline{\operatorname{ran}}\,\alpha = Y$ and since $\alpha \notin B$, Corollary 2.2 applies to $\alpha$ yielding $t\alpha = t\phi_{Y,Y} = t$ for all $t \in \operatorname{dom}\alpha$. Thus, $\alpha = \alpha^{-1}$ and $\iota = \alpha = \kappa$. $\square$

Yet another important consequence of Corollary 2.2 is the following

**Lemma 2.4.** *For each $\alpha \in J(S\{\zeta_k\})$,*

$$\alpha^2 = \alpha^3. \tag{4}$$

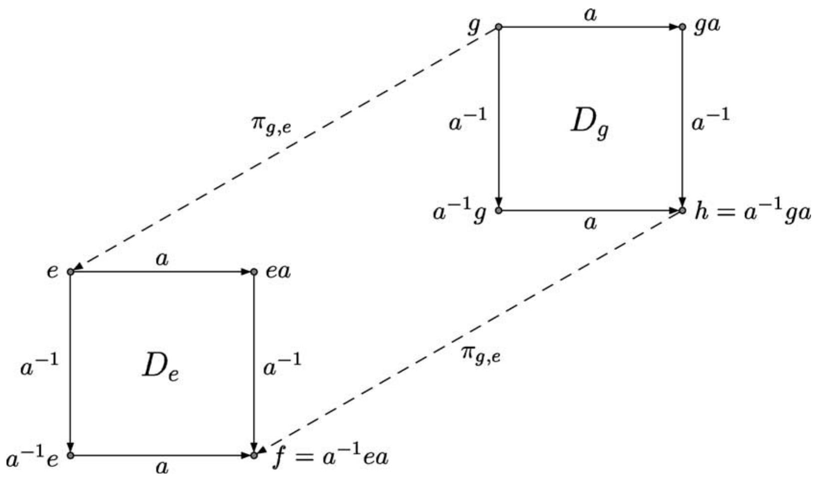


Fig. 4. The mapping $\pi_{g,e}$.

**Proof.** The claim is obvious if $\alpha$ belongs to $B$. Let $\alpha \in J(S\{\zeta_k\}) \setminus B$. If $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$ are disjoint, then $\alpha^2 = \varepsilon$ and $\alpha^2 = \alpha^3$. Otherwise $\underline{\operatorname{dom}}\,\alpha = \underline{\operatorname{ran}}\,\alpha = Y$ and by Corollary 2.2, $t\alpha = t\phi_{Y,Y} = t$ for all $t \in \operatorname{dom}\alpha$, whence $\alpha$ is an idempotent. □

Clearly, every inverse semigroup satisfying the identity (4) is combinatorial and completely semisimple.

Our main task in this section is to show that the semigroup $J(S\{\zeta_k\})$ belongs to the inverse semigroup variety $\mathbf{B_2^1}$ generated by the 6-element Brandt monoid $B_2^1$. We shall employ a deep result by Kad'ourek [8] who has provided an effective membership test for this variety. Let us introduce certain notions involved in Kad'ourek's test.

If $D$ is a subset of a semigroup $I$, we denote by $E(D)$ the set of all idempotents of $I$ contained in $D$; in other words, $E(D) = D \cap E(I)$. For $e, f \in E(I)$, we write $e \leqslant f$ if $ef = fe = e$. Now let $I$ be a combinatorial completely semisimple inverse semigroup and let $e, g \in E(I)$ be such that $e \leqslant g$. For any idempotent $h$ in the $\mathscr{D}$-class $D_g$ of the idempotent $g$, there exists a unique element $a \in D_g$ with $g = aa^{-1}$, $h = a^{-1}a = a^{-1}ga$. This then determines a mapping $\pi_{g,e} : E(D_g) \to E(D_e)$ by

$$h\pi_{g,e} = a^{-1}ea = (ea)^{-1}(ea).$$

This mapping produces an idempotent $f \leqslant h$ whose position with respect to the "anchor" idempotent $e$ of $D_e$ precisely corresponds to the position $h$ has had with respect to the "anchor" $g$ of its $\mathscr{D}$-class, see Fig. 4 where horizontal and vertical arrows symbolize multiplying respectively on the right and on the left.

The set $I/\mathscr{D}$ of all $\mathscr{D}$-classes of the semigroup $I$ has a natural partial ordering defined as follows: for $C, D \in I/\mathscr{D}$,

$$D \leqslant C \quad \text{if and only if} \quad e \leqslant f \text{ for some } e \in E(D),\ f \in E(C).$$

Let $C, D$ be $\mathscr{D}$-classes of $I$ and $D \leqslant C$. We define two symmetric relations on $E(D)$. The first one—the *projection relation*—is defined by

$$\pi(C, D) = \bigl\{(h_1\pi_{g,e}, h_2\pi_{g,e}) \mid g, h_1, h_2 \in E(C),\ e \in E(D),\ e \leqslant g\bigr\}.$$

The second relation contains all pairs of idempotents in $E(D)$ which have a common upper bound in $E(C)$:

$$\rho(C, D) = \bigl\{(f_1, f_2) \in E(D) \mid f_1, f_2 \leqslant g \text{ for some } g \in E(C)\bigr\}.$$

For any $D \in I/\mathscr{D}$, we write $[D) = \{C \in I/\mathscr{D} \mid D \leqslant C\}$. A (possibly empty) subset $\mathscr{K}$ of $I/\mathscr{D}$ is called a *filter* if $[D) \subseteq \mathscr{K}$ for all $D \in \mathscr{K}$. Given $D \in I/\mathscr{D}$ and a filter $\mathscr{K} \subseteq [D)$, we denote by $\tau(\mathscr{K}, D)$ the transitive closure of the relation

$$\bigcup\bigl\{\pi(C_1, D) \mid C_1 \in \mathscr{K}\bigr\} \cup \bigcup\bigl\{\rho(C_2, D) \mid C_2 \notin \mathscr{K}\bigr\}.$$

We say that the filter $\mathscr{K}$ *separates* $e, f \in E(D)$ if $(e, f) \notin \tau(\mathscr{K}, D)$.

Now we are able to formulate Kad'ourek's result, see [8, Theorem 2.3]:

**Theorem 2.5.** *Let a combinatorial completely semisimple inverse semigroup $I$ satisfy the following condition*:

(∗) *for any $\mathscr{D}$-classes $C, D$ of $I$ and for any $e, f \in E(D)$ such that $e \leqslant g$ and $f \nleqslant g$ for some $g \in E(C)$, there exists a filter $\mathscr{K} \subseteq [D)$ which does not include $C$ and separates $e$ from $f$.*

*Then $I$ belongs to the inverse semigroup variety $\mathbf{B}_2^1$.*

In fact, Kad'ourek has proved that if the semigroup $I$ is at most countable, then the condition (∗) is not only sufficient but also necessary for $I$ to belong to $\mathbf{B}_2^1$.

**Proposition 2.6.** *The semigroup $J(S\{\zeta_k\})$ belongs to $\mathbf{B}_2^1$.*

**Proof.** By Lemma 2.4 the semigroup $J(S\{\zeta_k\})$ is combinatorial and completely semisimple. In view of Theorem 2.5, it remains to verify that the condition (∗) holds true for all $\mathscr{D}$-classes of this semigroup. Thus, we fix an arbitrary pair $C, D$ of $\mathscr{D}$-classes in $J(S\{\zeta_k\})$ such that there exist idempotents $\eta, \vartheta \in D$, $\gamma \in C$ satisfying $\eta \leqslant \gamma$ and $\vartheta \nleqslant \gamma$. Clearly, $\eta \neq \vartheta$. We should find a filter $\mathscr{K} \subseteq [D)$ which does not include $C$ and separates $\eta$ from $\vartheta$. Let $Y = \underline{\mathrm{dom}}\,\eta$ and $Z = \underline{\mathrm{dom}}\,\vartheta$. There are two possible cases.

*Case* 1: $Y \neq Z$. Then the empty filter $\mathscr{K}$ does the job. Indeed, $C \notin \mathscr{K}$ and the relation $\bigcup\{\pi(C_1, D) \mid C_1 \in \mathscr{K}\}$ is empty. The relation $\tau(\mathscr{K}, D)$ must then reduce to the transitive closure of $\bigcup\{\rho(C_2, D) \mid C_2 \in [D)\}$. However, the domain of every idempotent possessing a common upper bound with $\eta$ (respectively, $\vartheta$) is contained in $Y$ (respectively, $Z$), and no chain of $\rho(C_2, D)$-related idempotents starting with $\eta$ reaches $\vartheta$. Thus, $\mathscr{K}$ separates $\eta$ from $\vartheta$.

*Case* 2: $Y = Z$. In view of Lemma 2.3 this case is only possible if $D = B \setminus \{\varepsilon\}$. We define $\mathscr{K}$ as

$$\{D' \in [D) \mid D' \nleqslant C\}.$$

Clearly, $\mathscr{K}$ is a filter and $C \notin \mathscr{K}$. It remains to verify that $\mathscr{K}$ separates $\eta$ from $\vartheta$. We shall need an auxiliary notion which can be introduced as follows.

The map that assigns to each non-zero idempotent in $B$ its (singleton) domain is one-to-one; in other words, non-zero idempotents of $B$ can be encoded by elements of our base set $\overline{T}_{m,K}(X)$. In turn, each element $t \in \overline{T}_{m,K}(X)$ is uniquely determined by the subset $T \in \{X, X_{j,k}, X'_{j,k}\}$ to which $t$ belongs and by the ordinal number $n(t)$ of the position $t$ occupies in the well-ordered set $T$. If $\{t\} = \operatorname{dom}\theta$ for some $\theta \in E(D)$, we call the ordinal $n(t)$ the *ordinal label* of the idempotent $\theta$ and denote it by $n(\theta)$.

**Lemma 2.7.** *If $D < D'$ for a $\mathscr{D}$-class $D'$ of $J(S\{\zeta_k\})$, then the ordinal labels of any two $\pi(D', D)$-related idempotents of $D$ coincide.*

**Proof.** Recall that $\chi_1, \chi_2 \in E(D)$ are $\pi(D', D)$-related if and only if there exist $\upsilon_1, \upsilon_2, \delta \in E(D')$ and $\zeta \in E(D)$ such that $\zeta \leqslant \delta$ and $\chi_i = \upsilon_i \pi_{\delta,\zeta}$ $(i = 1, 2)$. According to the definition of the projection mapping $\pi_{\delta,\zeta}$, this means that

$$\chi_i = \alpha_i^{-1}\zeta\alpha_i \quad (i = 1, 2), \tag{5}$$

where the elements $\alpha_i \in D'$ are uniquely determined from the conditions

$$\delta = \alpha_i\alpha_i^{-1},\ \upsilon_i = \alpha_i^{-1}\alpha_i \quad (i = 1, 2).$$

Since $D' \neq D = B \setminus \{\varepsilon\}$, $\alpha_1, \alpha_2$ do not belong to $B$. By Corollary 2.2, these mappings extend to order isomorphisms. Therefore the equalities (5) ensure that the ordinal labels of the idempotents $\chi_1$ and $\chi_2$ are equal to the ordinal label of the idempotent $\zeta$. □

Returning to the proof of our proposition, we observe that since inequalities between idempotent mappings correspond to containments between their domains, the conditions $\eta \leqslant \gamma$ and $\vartheta \nleqslant \gamma$ mean that $\operatorname{dom}\eta \subseteq \operatorname{dom}\gamma$ and $\operatorname{dom}\vartheta \nsubseteq \operatorname{dom}\gamma$, respectively. We denote by $\Gamma$ the set of the ordinals $\{n(y) \mid y \in \operatorname{dom}\gamma\}$. Then the ordinal label $n(\eta)$ belongs to this set, while the ordinal label $n(\vartheta)$ does not (because of $\underline{\operatorname{dom}}\,\vartheta = \underline{\operatorname{dom}}\,\eta = Y$). Therefore in order to show that the filter $\mathscr{K}$ separates $\eta$ from $\vartheta$, it suffices to prove that the ordinal label of every idempotent $\omega \in D$ which is $\tau(\mathscr{K}, D)$-related to an idempotent $\psi \in D$ with $n(\psi) \in \Gamma$ stays within the set $\Gamma$. According to the definition of the relation $\tau(\mathscr{K}, D)$, this amounts to considering the situations when either $(\psi, \omega) \in \pi(C_1, D)$ for some $C_1 \in \mathscr{K}$ or $(\psi, \omega) \in \rho(C_2, D)$ for some $C_2 \notin \mathscr{K}$. In the first situation, $D < C_1$ (because $D \notin \mathscr{K}$), hence Lemma 2.7 applies yielding $n(\omega) = n(\psi) \in \Gamma$. Thus, we may assume that the second situation takes place and that $\psi \neq \omega$.

By the construction of our filter, $C_2 \notin \mathscr{K}$ means that $C_2 \leqslant C$, that is, $\mu \leqslant \nu$ for some $\mu \in E(C_2)$, $\nu \in E(C)$. Applying the projection mapping $\pi_{\nu,\mu}$ to the idempotent $\gamma \in C$,

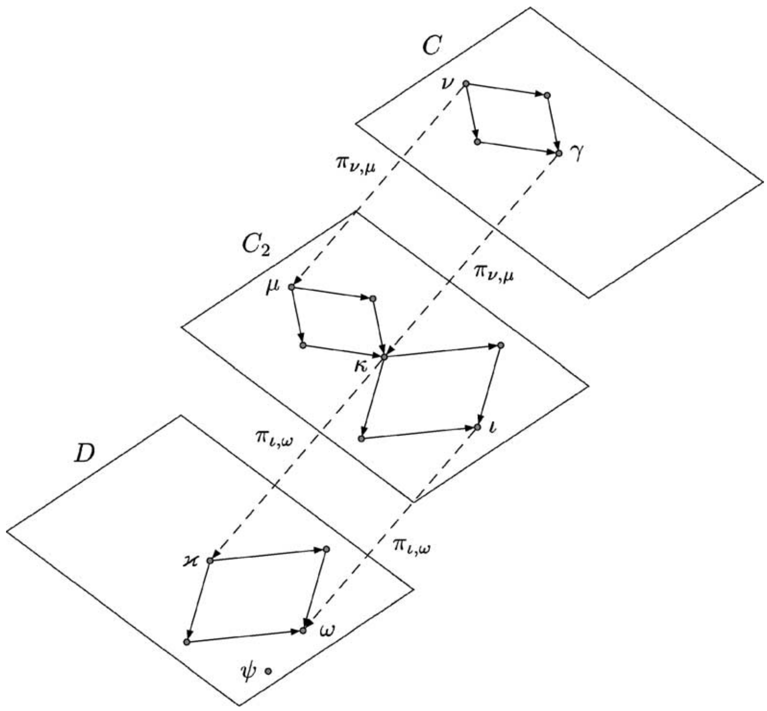


Fig. 5. Idempotents occurring in the proof that $n(\omega) = n(\psi)$.

we obtain an idempotent $\kappa \in C_2$ such that $\kappa \leqslant \gamma$. Since $(\psi, \omega) \in \rho(C_2, D)$, there exists an idempotent $\iota \in C_2$ such that $\psi, \omega \leqslant \iota$. Observe that $C_2 \neq D$—indeed, distinct idempotents within $D$ are incomparable, and therefore, $\psi$ and $\omega$ cannot have a common upper bound there. Now we apply the projection mapping $\pi_{\iota,\omega}$ to the idempotent $\kappa$. We obtain an idempotent $\varkappa \leqslant \kappa$. (Figure 5 should help the reader to keep track of the relations between various idempotents that occur in this proof.) Since $\kappa \leqslant \gamma$, we may conclude that $\varkappa \leqslant \gamma$ whence $n(\varkappa) \in \Gamma$. On the other hand, it is clear that $\iota\pi_{\iota,\omega} = \omega$ whence the idempotents $\omega$ and $\varkappa$ are $\pi(C_2, D)$-related. By Lemma 2.7, $n(\omega) = n(\varkappa)$ and $n(\omega) \in \Gamma$, as required. □

From Proposition 2.6 we immediately obtain

**Corollary 2.8.** *Each $m$-generated subsemigroup of the semigroup $S_m(U)$ embeds in an inverse semigroup which belongs to the variety $\mathbf{B}_2^1$.*

## 3. Semigroup quasiidentities of inverse semigroups

We recall some basic facts concerning semigroup quasiidentities and quasivarieties. A *semigroup quasiidentity* is an expression of the form

$$u_1 = v_1 \;\&\; u_2 = v_2 \;\&\; \cdots \;\&\; u_n = v_n \quad \Longrightarrow \quad u = v, \tag{6}$$

where $u_1, v_1, u_2, v_2, \dots, u_n, v_n, u, v$ are words over a countable infinite alphabet $A$. As usual, we denote by $A^+$ the free semigroup over $A$. A semigroup $S$ is said to *satisfy* the quasiidentity (6) if for every homomorphism $\varphi : A^+ \to S$, $u\varphi = v\varphi$ provided that $u_1\varphi = v_1\varphi$, $u_2\varphi = v_2\varphi, \dots, u_n\varphi = v_n\varphi$. A *semigroup quasivariety* is any class $\mathcal{Q}$ of semigroups defined by a set (say, $\Sigma$) of quasiidentities in the following sense: $\mathcal{Q}$ consists of all semigroups that satisfy all quasiidentities in $\Sigma$. The set $\Sigma$ is then said to constitute a *quasiidentity basis* for $\mathcal{Q}$; if $\mathcal{Q}$ possesses a finite basis, it is called a *finitely based* quasivariety.

For a class $\mathcal{C}$ of semigroups, $\operatorname{qvar} \mathcal{C}$ denotes the quasivariety *generated by* $\mathcal{C}$, that is, the least quasivariety containing $\mathcal{C}$. It is known that $\operatorname{qvar} \mathcal{C} = \mathbb{SPP}_U(\mathcal{C})$, where $\mathbb{S}$, $\mathbb{P}$ and $\mathbb{P}_U$ stand for the operators for formation of subsemigroups, direct products and ultraproducts respectively (cf., e.g., [3, Section 2.3]). The latter operator is perhaps less familiar than the other two but we do not reproduce its (somewhat bulky) definition because the only property of ultraproducts which we need in the present paper is that an ultraproduct of a family of semigroups is a homomorphic image of the direct product of this family.

In a similar way one defines quasiidentities and quasivarieties of inverse semigroups considered as algebras of type $\langle 2, 1\rangle$; of course, the role of $A^+$ is then played by the free inverse semigroup $FI(A)$ over the alphabet $A$. If $\mathcal{C}$ is a class of inverse semigroups, then $\operatorname{qvar}_{\mathrm{inv}} \mathcal{C}$ denotes the inverse semigroup quasivariety generated by $\mathcal{C}$. The relation between $\operatorname{qvar}_{\mathrm{inv}} \mathcal{C}$ and $\operatorname{qvar} \mathcal{C}$ is easy to see:

**Lemma 3.1.** *For an arbitrary class $\mathcal{C}$ of inverse semigroups, the quasivariety* $\operatorname{qvar} \mathcal{C}$ *coincides with the class of all subsemigroups of members of the inverse semigroup quasivariety* $\operatorname{qvar}_{\mathrm{inv}} \mathcal{C}$.

**Proof.** Every ultraproduct and every direct product of a family of inverse semigroups again is an inverse semigroup. Hence $\mathbb{PP}_U(\mathcal{C}) \subseteq \operatorname{qvar}_{\mathrm{inv}} \mathcal{C}$ and $\operatorname{qvar} \mathcal{C} = \mathbb{SPP}_U(\mathcal{C}) \subseteq \mathbb{S}(\operatorname{qvar}_{\mathrm{inv}} \mathcal{C})$. The converse inclusion is obvious. $\square$

We are interested in the finite basis problem for *semigroup* quasivarieties generated by inverse semigroups. It is well known that the answer to the problem may vary depending on the nature of the inverse generators. For example, by Maltsev's celebrated result ([11], see also [2, Chapter 12]), the semigroup quasivariety generated by the class of all groups, that is, the quasivariety of all group embeddable semigroups, is non-finitely based. Similarly, semigroup quasiidentities holding in all inverse semigroups are non-finitely based—this result is due to Schein ([13], see also [14]). On the other hand, the semigroup quasivariety generated by the class of all abelian groups is known to be finitely based: its quasiidentity basis consists of the commutative and the cancellation laws. Similarly, the quasivariety of all semigroups embeddable in commutative Clifford semigroups (that is, semilattices of abelian groups) is defined by the commutative and the separative laws. All of these important examples are special, and each of them has required finding an explicit (infinite or finite) list of embeddability conditions. In contrast, as the first application of our technique, we prove a fairly general result showing that inverse semigroups "almost always" generate non-finitely based semigroup quasivarieties. As the reader will see, our proof avoids any—explicit or implicit—appeal to embeddability criteria.

**Theorem 3.2.** *Let $\mathcal{C}$ be an arbitrary class of inverse semigroups such that the inverse semigroup quasivariety* $\mathrm{qvar}_{\mathrm{inv}}\,\mathcal{C}$ *contains the variety* $\mathbf{B}_2^1$ *but does not contain the free inverse semigroup $FI(A)$ over a countable, infinite alphabet $A$. Then the semigroup quasivariety* $\mathrm{qvar}\,\mathcal{C}$ *has no basis of quasiidentities involving finitely many variables and in particular has no finite quasiidentity basis.*

**Proof.** Using the Wagner–Preston representation, we faithfully represent the semigroup $FI(A)$ as a semigroup $U$ of partial mappings on a set $X$ generated (as a plain semigroup) by a certain set of extendable injections $\{b_k\}_{k\in K}$. Clearly $U$ coincides with its inverse hull in the symmetric inverse semigroup $\mathcal{I}_X$.

Now suppose that the semigroup quasivariety $\mathcal{Q} = \mathrm{qvar}\,\mathcal{C}$ has a basis $\Sigma$ consisting of quasiidentities which involve only finitely many variables, and let $m$ be the number of variables that occur in quasiidentities in $\Sigma$. Consider the semigroup $S_m(U)$ constructed as in Section 1. If $S_m(U)$ belongs to the quasivariety $\mathcal{Q}$, then by Lemma 3.1, $S_m(U)$ embeds into an inverse semigroup $I \in \mathrm{qvar}_{\mathrm{inv}}\,\mathcal{C}$. Proposition 1.4 then ensures that $U$ is a homomorphic image of an inverse subsemigroup $V$ of $I$. Since $U \cong FI(A)$ is the free inverse semigroup, this implies that it in fact embeds in $I$. Indeed, if we choose, for each free generator $a \in A$, a representative in the preimage of $a$ in $V$, then the inverse subsemigroup of $V$ generated by all these representatives is easily seen to be isomorphic to $FI(A)$. Hence $FI(A) \in \mathrm{qvar}_{\mathrm{inv}}\,\mathcal{C}$, a contradiction. Thus, $S_m(U)$ does not belong to $\mathcal{Q}$.

On the other hand, by Corollary 2.8, each $m$-generated subsemigroup $S$ of the semigroup $S_m(U)$ embeds in an inverse semigroup from the variety $\mathbf{B}_2^1 \subseteq \mathrm{qvar}_{\mathrm{inv}}\,\mathcal{C}$, and therefore, $S$ belongs to the quasivariety $\mathcal{Q}$. This means that $S$ satisfies all quasiidentities in $\Sigma$. However, when evaluating a quasiidentity in at most $m$ variables in the semigroup $S_m(U)$, we actually evaluate it in an $m$-generated subsemigroup of $S_m(U)$. Thus, the semigroup $S_m(U)$ also satisfies all quasiidentities in $\Sigma$, whence it must belong to $\mathcal{Q}$.

The conclusions of the two previous paragraphs contradict each other thus showing that, in fact, the quasivariety $\mathcal{Q}$ cannot be defined by quasiidentities involving only finitely many variables. □

We observe that the proof of Theorem 3.2 in fact applies to each semigroup quasivariety $\mathcal{Q}'$ such that $\mathrm{qvar}\,\mathbf{B}_2^1 \subseteq \mathcal{Q}' \subseteq \mathrm{qvar}\,\mathcal{C}$ (even if $\mathcal{Q}'$ is not generated by its inverse semigroups). In other words, the whole interval of the lattice of semigroup quasivarieties between the quasivarieties $\mathrm{qvar}\,\mathbf{B}_2^1$ and $\mathrm{qvar}\,\mathcal{C}$ consists of non-finitely based quasivarieties.

In order to demonstrate how weak are the conditions imposed on the class $\mathcal{C}$ in Theorem 3.2, consider them in the special case when $\mathcal{C}$ is a variety of inverse semigroups. Then we have

**Corollary 3.3.** *If* $\mathbf{V}$ *is a variety of inverse semigroups containing the variety* $\mathbf{B}_2^1$*, then the semigroup quasivariety* $\mathrm{qvar}\,\mathbf{V}$ *is non-finitely based.*

**Proof.** If $\mathbf{V} = \mathbf{Inv}$, the variety of all inverse semigroups, then the claim follows from the aforementioned result by Schein [13]. Otherwise $\mathbf{V}$ does not contain the free inverse semigroup over a countable infinite alphabet, and Theorem 3.2 applies.

If an inverse semigroup variety $\mathbf{V}$ does not contain the variety $\mathbf{B}_2^1$, then the structure of $\mathbf{V}$ and its position within the lattice $L(\mathbf{Inv})$ of all inverse semigroup varieties are fairly well understood (see [12, §XII.4]). Namely, $\mathbf{V}$ consists of so-called strict inverse semigroups and belongs to one of the three isomorphic bottom layers of the lattice $L(\mathbf{Inv})$ (the three layers consist of varieties of groups, non-group varieties of Clifford semigroups and non-Clifford varieties generated by Brandt semigroups). In particular, if we restrict to varieties of combinatorial inverse semigroups, there are only three varieties to which Corollary 3.3 does not apply.

## 4. A generalization: semigroup quasiidentities of locally $E$-solid regular semigroups

We want to demonstrate how Corollary 3.3 can be extended to certain classes of regular semigroups beyond the variety of inverse semigroups. Recall that an *e-variety* is a class of regular semigroups that is closed under the taking of direct products, regular subsemigroups and homomorphic images. This fruitful concept was introduced by Hall [5] and—in the context of orthodox semigroups—by Kad'ourek and Szendrei [10]. Many important classes of regular semigroups are known to constitute $e$-varieties; for instance, the classes **Orth** of all orthodox semigroups and **ES** of all $E$-solid regular semigroups are $e$-varieties (recall that a semigroup $S$ is said to be *$E$-solid* if for all idempotents $e, f, g \in E(S)$ with $e \mathscr{L} f \mathscr{R} g$, there exists an idempotent $h \in E(S)$ such that $e \mathscr{R} h \mathscr{L} g$). We also note that for the inverse case, $e$-varieties are precisely inverse semigroup varieties of type $\langle 2, 1\rangle$ as considered in Section 3.

Given an $e$-variety $\mathbf{V}$, we denote by $\operatorname{Loc}\mathbf{V}$ the class of all regular semigroups $S$ whose *local submonoids* $eSe$, $e \in E(S)$, belong to $\mathbf{V}$; it is well known (and easy to see) that $\operatorname{Loc}\mathbf{V}$ is again an $e$-variety. In this way we obtain the $e$-varieties $\operatorname{Loc}\mathbf{Inv}$ of all *locally inverse semigroups*, $\operatorname{Loc}\mathbf{Orth}$ of all *locally orthodox semigroups*, $\operatorname{Loc}\mathbf{ES}$ of all regular *locally $E$-solid semigroups*, etc. Without going into detail, we note that the importance of the $e$-variety $\operatorname{Loc}\mathbf{ES}$ has been revealed by recent results by Kad'ourek [9] and Churchill and the third-named author [1] who have discovered that $\operatorname{Loc}\mathbf{ES}$ is the largest $e$-variety of regular semigroups which possesses so-called trifree objects—these are generalizations of free objects in (conventional) varieties. Thus, in a certain sense $\operatorname{Loc}\mathbf{ES}$ is the largest “tractable” $e$-variety; besides that, it contains all concrete $e$-varieties that have ever occurred in the literature. We are going to generalize Corollary 3.3 to $e$-subvarieties of $\operatorname{Loc}\mathbf{ES}$.

First, we extend Proposition 1.4 to locally $E$-solid regular semigroups. Take an arbitrary semigroup $U$ generated by a family $\{b_k\}_{k\in K}$ of partial injective mappings on a set $X$ and construct the semigroup $S_m(U)$ as described in Section 1. Let $S_m^1(U)$ denote the semigroup $S_m(U)$ with the identity mapping on the set $\overline{T}_{m,K}(X)$ adjoined. Clearly, Propositions 1.1 and 1.2 (that we stated for $S_m(U)$) remain valid for $S_m^1(U)$ as well.

**Proposition 4.1.** *If the semigroup $S_m^1(U)$ embeds in a locally $E$-solid regular semigroup $T$, then the inverse hull $J(U)$ of the semigroup $U$ belongs to the $e$-variety generated by $T$.*

**Proof.** We shall employ the following property of locally $E$-solid regular semigroups:

**Lemma 4.2** [4, Theorem 7]. *On each locally $E$-solid regular semigroup $T$ there exists a least congruence $\vartheta$ such that $T/\vartheta$ is a locally inverse semigroup, and for each idempotent $e \in E(T)$, the $\vartheta$-class $e^{\vartheta}$ is a completely simple subsemigroup of $T$.*

We identify the semigroup $S_m^1(U)$ with its image in $T$. Our first goal is to show that the least locally inverse congruence $\vartheta$ on $T$ separates elements of $S_m^1(U)$. Arguing by contradiction, we assume that the restriction $\sigma$ of $\vartheta$ to $S_m^1(U)$ is a non-trivial congruence on that semigroup. Then by Proposition 1.2, $\sigma$ contains the Rees congruence on $S_m^1(U)$ corresponding to the Brandt ideal $B$, in particular, $B$ is contained in a single $\sigma$-class. By Lemma 4.2 the $\vartheta$-class containing this $\sigma$-class must form a completely simple subsemigroup in $T$. However, $B$ has at least $2m+1$ idempotents which all commute while no completely simple semigroup can have a pair of distinct commuting idempotents, a contradiction.

We have thus proved that if the semigroup $S_m^1(U)$ embeds in a locally $E$-solid regular semigroup $T$, then $S_m^1(U)$ may be thought of as a subsemigroup in the locally inverse semigroup $S = T/\vartheta$. Let $\iota$ denote the identity element of $S_m^1(U)$; then $S_m^1(U)$ is contained in the local submonoid $\iota S\iota$ which is already an inverse semigroup. We are in a position to apply Proposition 1.4 concluding that $J(U)$ is an inverse divisor of $\iota S\iota$. Hence $J(U)$ belongs to the inverse semigroup variety generated by the semigroup $\iota S\iota$ which—as a regular subsemigroup of a homomorphic image of $T$—in turn belongs to the $e$-variety generated by $T$. □

Now we can prove the promised generalization of Corollary 3.3.

**Theorem 4.3.** *Let* $\mathbf{V}$ *be an $e$-variety of locally $E$-solid regular semigroups that contains the inverse semigroup variety* $\mathbf{B}_2^1$ *but does not contain the variety of all inverse semigroups. Then the semigroup quasivariety* qvar $\mathbf{V}$ *is non-finitely based.*

**Proof.** Let $U$ denote an inverse semigroup of partial injections which does not belong to $\mathbf{V}$ (for instance, one can take $U = FI(A)$ as in the proof of Theorem 3.2). Then, for each positive integer $m$, we consider the semigroup $S_m^1(U)$.

By definition, every $e$-variety is closed under the taking of homomorphic images and direct products, whence $e$-varieties are also closed under ultraproducts. Therefore the formula qvar $\mathbf{V} = \mathbb{SPP}_U(\mathbf{V})$ reduces to qvar $\mathbf{V} = \mathbb{S}(\mathbf{V})$; in other words, if the semigroup $S_m^1(U)$ lies in the quasivariety $\mathcal{Q} = \text{qvar}\,\mathbf{V}$, then this semigroup embeds in a locally $E$-solid regular semigroup $T \in \mathbf{V}$. By Proposition 4.1, this implies that $U$ (which of course coincides with its own inverse hull) belongs to the $e$-variety generated by $T$ and hence to the $e$-variety $\mathbf{V}$, a contradiction to the choice of $U$. Thus, $S_m^1(U) \notin \mathcal{Q}$.

It is easy to see that if an inverse semigroup variety $\mathbf{M}$ is generated by a monoid, then for each inverse semigroup $S \in \mathbf{M}$, we also have $S^1 \in \mathbf{M}$. Further, every $m$-generated subsemigroup $P$ of the semigroup $S_m^1(U)$ either is a subsemigroup in $S_m(U)$ or can be represented as $Q^1$ for some $(m-1)$-generated subsemigroup $Q$ in $S_m(U)$. Combining these observations with Corollary 2.8, we readily deduce that $P$ embeds in an inverse semigroup from the variety $\mathbf{B}_2^1 \subseteq \mathbf{V}$. Thus, $P$ belongs to $\mathcal{Q}$.

We have proved that for any positive integer $m$, the semigroup $S_m^1(U)$ does not lie in the quasivariety $\mathcal{Q}$ while every $m$-generated subsemigroup of $S_m^1(U)$ belongs to $\mathcal{Q}$. The argument concludes as in Theorem 3.2. $\square$

We conclude with a remark similar to that made after Theorem 3.2: it is easy to see that the proof of Theorem 4.3 applies to every semigroup quasivariety $\mathcal{Q}'$ (not necessarily generated by regular semigroups) such that qvar $\mathbf{B_2^1} \subseteq \mathcal{Q}' \subseteq$ qvar $\mathbf{V}$.

## References

[1] G.A. Churchill, P.G. Trotter, A unified approach to biidentities for $e$-varieties, Semigroup Forum 60 (2000) 208–230.
[2] A.H. Clifford, G.B. Preston, The Algebraic Theory of Semigroups, Vol. I, II, Am. Math. Soc., Providence, 1961, 1967.
[3] V.A. Gorbunov, Algebraic Theory of Quasivarieties, Plenum, New York, 1998.
[4] T.E. Hall, Some properties of local subsemigroups inherited by larger subsemigroups, Semigroup Forum 25 (1982) 35–49.
[5] T.E. Hall, Identities for existence varieties of regular semigroups, Bull. Austral. Math. Soc. 40 (1989) 59–77.
[6] P.M. Higgins, S.W. Margolis, Finite aperiodic semigroups with commuting idempotents and generalizations, Israel J. Math. 116 (2000) 367–380.
[7] J.M. Howie, Fundamentals of Semigroup Theory, 2nd Edition, Clarendon, Oxford, 1995.
[8] J. Kad'ourek, On varieties of combinatorial inverse semigroups I, Semigroup Forum 43 (1991) 305–330.
[9] J. Kad'ourek, On some existence varieties of locally orthodox semigroups, Internat. J. Algebra Comput. 7 (1997) 93–131.
[10] J. Kad'ourek, M.B. Szendrei, A new approach in the theory of orthodox semigroups, Semigroup Forum 40 (1990) 257–296.
[11] A.I. Maltsev, On inclusion of associative systems into groups II, Mat. Sb. 8 (1940) 251–264 (in Russian).
[12] M. Petrich, Inverse Semigroups, Wiley, New York, 1984.
[13] B.M. Schein, Embedding semigroups into inverse semigroups, in: Algebra and Number Theory, Vol. 2, Nalchik, 1977, pp. 147–163 (in Russian).
[14] B.M. Schein, Subsemigroups of inverse semigroups, Matematiche 51 (1996) 205–227.
[15] P.G. Trotter, $E$-varieties of regular semigroups, in: J. Almeida, G.M.S. Gomes, P.V. Silva (Eds.), Semigroups, Automata and Languages, World Scientific, Singapore–New Jersey–London–Hong Kong, 1996, pp. 247–262.